

Dilatively stable stochastic processes and aggregate similarity

MÁTYÁS BARCZY^{*, \diamond} , PETER KERN^{**}, GYULA PAP^{***}

* Faculty of Informatics, University of Debrecen, Pf. 12, H-4010 Debrecen, Hungary.

** Mathematical Institute, Heinrich-Heine-University Düsseldorf, Universitätsstr. 1, D-40225 Düsseldorf, Germany.

*** Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary.

e-mails: barczy.matyas@inf.unideb.hu (M. Barczy), Peter.Kern@uni-duesseldorf.de (P. Kern), papgy@math.u-szeged.hu (G. Pap).

\diamond Corresponding author.

Abstract

Dilatively stable processes generalize the class of infinitely divisible self-similar processes. We reformulate and extend the definition of dilative stability introduced by Iglói (2008) using characteristic functions. We also generalize the concept of aggregate similarity introduced by Kaj (2005). It turns out that these two notions are essentially the same for infinitely divisible processes. Examples of dilatively stable generalized fractional Lévy processes are given and we point out that certain limit processes in aggregation models are dilatively stable.

1 Dilative stability and aggregate similarity

Self-similarity is a scaling property of stochastic processes. It was Lamperti's paper [13] which called the attention to the significance of this property (named there semi-stability). As a generalization, Iglói [7, Definition 2.1.3 and Theorem 2.2.1] introduced a more general scaling property of certain infinitely divisible processes called dilative stability. Iglói [7, Examples 2.1.5-7] already provided some important (non self-similar) dilatively stable processes, such as non-Gaussian moving-average fractional Lévy motions. Roughly speaking, non-Gaussian fractional Lévy motions are not self-similar but they belong to a wider class of processes, to the class of dilatively stable processes, which underlines the importance of dilative stability. To put dilative stability into a more general context, we note that there are many other known examples

2010 Mathematics Subject Classifications: 60G18, 60G22, 39B05.

Key words and phrases: dilatively stable process, self-similar process, fractional Lévy motion, aggregate similarity.

The research has been supported by the DAAD-MÖB Research Grant No. 55757 partially financed by the German Federal Ministry of Education and Research (BMBF).

of stochastic processes that do not fit into the scope of self-similarity, but are very natural from the point of view of scaling limit theorems and various types of invariance properties in probability theory, besides Iglói [7], see, e.g., Biermé et al. [2], Kaj [9] and Pilipajnsjautė and Surgailis [15]. However, there is no common framework for these kinds of processes. The examples given by Iglói [7] and the new examples presented in Sections 2 and 3 show that the class of dilatively stable processes is not only much wider than that of self-similar processes, but it contains several important non self-similar processes. So the systematic study of dilative stability started by Iglói [7] deserves to be continued.

Further, Kaj [9, Section 3.6] introduced a similar concept called aggregate similarity appearing for certain heavy-tailed limit processes in teletraffic models under an intermediate growth condition.

This paper has three-fold aims. First, we reformulate and extend the definition of dilative stability introduced by Iglói [7] using characteristic functions. Next, we generalize the concept of aggregate similarity due to Kaj, pointing out that these generalized aggregate similarity and dilative stability are essentially the same for infinitely divisible processes. Finally, we present new examples of dilatively stable processes that are not self-similar.

In what follows, \mathbb{N} , \mathbb{R}_+ , \mathbb{R}_{++} and \mathbb{R} will denote the set of positive integers, non-negative real numbers, positive real numbers and real numbers, respectively. Further, let \mathbb{T} be either \mathbb{R} , \mathbb{R}_+ or \mathbb{R}_{++} . All the stochastic processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

First, we give both definitions: self-similarity and dilative stability.

1.1 Definition. *Let $\alpha > 0$. A process $(X_t)_{t \in \mathbb{T}}$ is called α -self-similar if in terms of characteristic functions it fulfills the space-time scaling relation*

$$(1.1) \quad \mathbb{E} \left(e^{i \sum_{j=1}^k \theta_j X_{T t_j}} \right) = \mathbb{E} \left(e^{i T^\alpha \sum_{j=1}^k \theta_j X_{t_j}} \right)$$

for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, $t_1, \dots, t_k \in \mathbb{T}$.

If $0 \in \mathbb{T}$, then $\mathbb{P}(X_0 = 0) = 1$, see, e.g., Samorodnitsky and Taqqu [16, page 312]. Note that property (1.1) is equivalent to

$$(1.2) \quad \forall T > 0 : X(T \cdot) \stackrel{\text{fd}}{\sim} T^\alpha X(\cdot),$$

where $\stackrel{\text{fd}}{\sim}$ denotes that the finite-dimensional distributions are the same. If a stochastically continuous process $(X_t)_{t \in \mathbb{R}_+}$ is f -self-similar for some function $f : (0, \infty) \rightarrow (0, \infty)$ in the sense that it satisfies (1.1) replacing T^α by $f(T)$ on the right-hand side of (1.1), then there exists an $\alpha \geq 0$ such that $f(T) = T^\alpha$, $T > 0$, and in case of $\alpha = 0$, one has $\mathbb{P}(X_t = X_0) = 1$, $t \in \mathbb{R}_+$, see Lamperti [13, Theorem 1]. In the definition of self-similarity the case $\alpha = 0$ is usually excluded, such as in Definition 1.1.

Dilative stability is an analogous property of certain infinitely divisible processes (all finite-dimensional distributions are infinitely divisible) involving a scaling also in the convolution exponent.

For an infinitely divisible process $(X_t)_{t \in \mathbb{T}}$ and for $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{T}$, let $\psi_{t_1, \dots, t_k} : \mathbb{R}^k \rightarrow \mathbb{C}$ denote the characteristic exponent of $(X_{t_1}, \dots, X_{t_k})$, i.e., the unique continuous function such that $\psi_{t_1, \dots, t_k}(0, \dots, 0) = 0$ and

$$\mathbb{E} \left(e^{i \sum_{j=1}^k \theta_j X_{t_j}} \right) = e^{-\psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k)} \quad \text{for all } \theta_1, \dots, \theta_k \in \mathbb{R},$$

see, e.g., (the multi-dimensional version of) Lemma 3.2.11 in Stroock [18].

1.2 Definition. Let $\alpha, \delta \in \mathbb{R}$. An infinitely divisible process $(X_t)_{t \in \mathbb{T}}$ is said to be (α, δ) -dilatively stable if the scaling relation

$$(1.3) \quad \psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k) = T^\delta \psi_{t_1, \dots, t_k}(T^{\alpha - \frac{\delta}{2}} \theta_1, \dots, T^{\alpha - \frac{\delta}{2}} \theta_k)$$

holds for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, $t_1, \dots, t_k \in \mathbb{T}$.

The functional equation (1.3) defines dilative stability in an algebraic sense. A probabilistic interpretation is given in Remark 1.3 below. Note that a process $(X_t)_{t \in \mathbb{T}}$ fulfilling the scaling relation (1.3) with $\alpha > 0$ and $\delta = 0$ is an α -self-similar process. Hence, for infinitely divisible processes, $(\alpha, 0)$ -dilative stability is just α -self-similarity. Hence dilative stability is a generalization of self-similarity. We call the attention that under the conditions of Definition 1.2, the parameters α and δ are not unique in general, see Example 2.7, or the simple case of symmetric stable processes given below.

To give a very first elementary example for a dilatively stable process, we point out that every Lévy process $(L_t)_{t \in \mathbb{R}_+}$ is $(\frac{1}{2}, 1)$ -dilatively stable. Indeed, for all $k \in \mathbb{N}$, $0 \leq t_1 < t_2 < \dots < t_k$ and $T > 0$, the increments $L_{Tt_j} - L_{Tt_{j-1}}$, $j \in \{1, \dots, k\}$, are independent, and for each $j \in \{1, \dots, k\}$, the distribution of $L_{Tt_j} - L_{Tt_{j-1}}$ is the T -th convolution power of the distribution of $L_{t_j} - L_{t_{j-1}}$, and hence the distribution of $(L_{Tt_1}, L_{Tt_2} - L_{Tt_1}, \dots, L_{Tt_k} - L_{Tt_{k-1}})$ is the T -th convolution power of the distribution of $(L_{t_1}, L_{t_2} - L_{t_1}, \dots, L_{t_k} - L_{t_{k-1}})$. Consequently, the distribution of $(L_{Tt_1}, L_{Tt_2}, \dots, L_{Tt_k})$ is the T -th convolution power of the distribution of $(L_{t_1}, L_{t_2}, \dots, L_{t_k})$. In the language of Definition 1.2, this means that $\psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k) = T \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k)$.

In particular, if $(L_t)_{t \in \mathbb{R}_+}$ is a symmetric $\frac{1}{H}$ -stable Lévy process with Hurst index $H \geq \frac{1}{2}$, then the above considerations yield that the process $(L_t)_{t \in \mathbb{R}_+}$ is $(\frac{1}{2}, 1)$ -dilatively stable. Since the process $(L_t)_{t \in \mathbb{R}_+}$ is H -self-similar, it is $(H, 0)$ -dilatively stable as well. In fact, $(L_t)_{t \in \mathbb{R}_+}$ is (α, δ) -dilatively stable for all $\alpha, \delta \in \mathbb{R}$ satisfying $\delta + (\alpha - \frac{\delta}{2}) \frac{1}{H} = 1$. Indeed, for each $t \geq 0$, the characteristic exponent of L_t has the form $\psi_t(\theta) = ct|\theta|^{\frac{1}{H}}$, $\theta \in \mathbb{R}$, for some $c > 0$, hence for all $T > 0$, we have $\psi_{Tt}(\theta) = cTt|\theta|^{\frac{1}{H}} = cT^\delta t |T^{\alpha - \frac{\delta}{2}} \theta|^{\frac{1}{H}} = T^\delta \psi_t(T^{\alpha - \frac{\delta}{2}} \theta)$, and independence of the increments yields the scaling relation (1.3). This is an example of a process that is (α, δ) -dilatively stable with infinitely many choices of (α, δ) , and α and δ can be negative as well.

1.3 Remark. For historical fidelity, we note that Iglói [7] originally formulated the scaling property (1.3) in the language of equality of finite-dimensional distributions, namely,

$$(1.4) \quad \forall T > 0 : X(T \cdot) \stackrel{\text{fd}}{\sim} T^{\alpha - \frac{\delta}{2}} X^{\otimes T^\delta}(\cdot),$$

where for all $c > 0$, we denote by $X^{\otimes c}$ the c -th convolution power of X , that is, $X^{\otimes c}$ is a process whose finite-dimensional distributions are the c -th convolution powers of the corresponding ones of X . Comparing the scaling relations (1.2) and (1.4), one can realize that in case of a dilatively stable process X , the time scaled process $(X_{Tt})_{t \in \mathbb{T}}$ coincides, in the sense of finite dimensional distributions, with X appropriately scaled in space and in convolution exponent as well, whereas in case of self-similarity, only some space scaling is allowed. Roughly speaking, scaling in the convolution exponent is the main new ingredient of dilative stability compared to self-similarity.

Iglói [7] defined dilative stability under some additional assumptions as given in Theorem 1.7 below. From this point of view, our Definition 1.2 is more general. Note that the condition $X_0 = 0$ in Theorem 1.7 turns out to be not so restrictive, see Iglói and Barczy [8, Appendix A] for details. \square

As it was mentioned, Kaj [9] introduced the notion of aggregate similarity. Namely, by Definition 1 in Kaj [9], a stochastic process $(X_t)_{t \in \mathbb{T}}$ is called aggregate similar with parameter $\varrho \in \mathbb{R}$, if it fulfills the scaling relation

$$(1.5) \quad \sum_{i=1}^m X^{(i)}(\cdot) \stackrel{\text{fd}}{\sim} m^\varrho X(m^{-\varrho} \cdot) \quad \text{for all } m \in \mathbb{N},$$

where $(X_t^{(i)})_{t \in \mathbb{T}}$, $i \in \mathbb{N}$, are independent identically distributed copies of $(X_t)_{t \in \mathbb{T}}$. The parameter ϱ is called rigidity index. Kaj additionally assumes that $\mathbb{T} = \mathbb{R}_+$, $(X_t)_{t \in \mathbb{T}}$ has stationary increments and fulfills $X_0 = 0$, $\mathbb{E}(X_t) = 0$ for all $t \in \mathbb{T}$, but these assumptions are not important for our approach. We introduce the notion of aggregate similarity with parameters $\varrho_1, \varrho_2 \in \mathbb{R}$.

1.4 Definition. Let $\varrho_1, \varrho_2 \in \mathbb{R}$. A process $(X_t)_{t \in \mathbb{T}}$ is called (ϱ_1, ϱ_2) -aggregate similar if it fulfills the scaling relation

$$(1.6) \quad \sum_{i=1}^m X^{(i)}(\cdot) \stackrel{\text{fd}}{\sim} m^{\varrho_1} X(m^{-\varrho_2} \cdot) \quad \text{for all } m \in \mathbb{N},$$

where $(X_t^{(i)})_{t \in \mathbb{T}}$, $i \in \mathbb{N}$, are independent identically distributed copies of $(X_t)_{t \in \mathbb{T}}$.

Under mild regularity assumptions, (1.6) already implies the scaling relation (1.3) of dilative stability as follows.

- 1.5 Proposition.** (i) If $(X_t)_{t \in \mathbb{T}}$ is a (α, δ) -dilatively stable process for some $(\alpha, \delta) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, then it is $(\frac{1}{2} - \frac{\alpha}{\delta}, -\frac{1}{\delta})$ -aggregate similar.
- (ii) If a process $(X_t)_{t \in \mathbb{T}}$ is infinitely divisible, its finite dimensional distributions are weakly right-continuous, and it is (ϱ_1, ϱ_2) -aggregate similar for some $(\varrho_1, \varrho_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, then it is $(\frac{\varrho_1}{\varrho_2} - \frac{1}{2\varrho_2}, -\frac{1}{\varrho_2})$ -dilatively stable.

The proof of Proposition 1.5 is given in Section 4. So for infinitely divisible processes, dilative stability and aggregate similarity, given in Definitions 1.2 and 1.4, are essentially the same. But note that the purely self-similar case is ruled out in Proposition 1.5, since $(\alpha, 0)$ -dilative stability cannot be handled.

Next we introduce a notion of (f, g) -dilative stability in an analogy with f -self-similarity.

1.6 Definition. Let $f, g : (0, \infty) \rightarrow (0, \infty)$ be continuous functions. An infinitely divisible process $(X_t)_{t \in \mathbb{T}}$ is said to be (f, g) -dilatively stable if the scaling relation

$$(1.7) \quad \psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k) = g(T)\psi_{t_1, \dots, t_k}(f(T)\theta_1, \dots, f(T)\theta_k)$$

holds for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, $t_1, \dots, t_k \in \mathbb{T}$.

Theorem 2.2.1 and Proposition 2.1.4 in Iglói [7] imply that, under some additional assumptions, the functions f and g are uniquely determined power functions.

1.7 Theorem. (Iglói [7]) Let $(X_t)_{t \in \mathbb{R}_+}$ be an (f, g) -dilatively stable process. Suppose that $X_0 = 0$, $(X_t)_{t \in \mathbb{R}_+}$ is not the identically zero process, $(X_t)_{t \in \mathbb{R}_+}$ is right continuous in distribution, X_1 is non-Gaussian and X_t has finite moments of all orders for all $t \in \mathbb{R}_+$, depending right continuously on t . Then there exist unique $\alpha \in (0, \infty)$ and $\delta \in [2\alpha, \infty)$ such that $f(T) = T^{\alpha - \frac{\delta}{2}}$, $T > 0$, and $g(T) = T^\delta$, $T > 0$.

Next we give further sufficient conditions under which the functions f and g must be uniquely determined power functions.

1.8 Theorem. Let $(X_t)_{t \in \mathbb{T}}$ be an (f, g) -dilatively stable process.

- (i) If X is not the identically zero process and $f(T) = T^\beta$ for some $\beta \in \mathbb{R}$ and all $T > 0$, then there exists a unique $\gamma \in \mathbb{R}$ such that $g(T) = T^\gamma$, $T > 0$.
- (ii) If X_1 is not Gaussian, but has a Gaussian component in its Lévy–Khintchine representation, then there exist unique $\beta, \gamma \in \mathbb{R}$ such that $f(T) = T^\beta$, $T > 0$, and $g(T) = T^\gamma$, $T > 0$.
- (iii) If $\mathbb{E}(X_1^2) < \infty$, $\mathbb{E}(X_1) \neq 0$ and $\text{Var}(X_1) > 0$, then there exist unique $\beta, \gamma \in \mathbb{R}$ such that $f(T) = T^\beta$, $T > 0$, and $g(T) = T^\gamma$, $T > 0$.
- (iv) If X_1 is not Gaussian, $\mathbb{E}(X_1^2) < \infty$ and $\text{Var}(X_1) > 0$, then there exist unique $\beta, \gamma \in \mathbb{R}$ such that $f(T) = T^\beta$, $T > 0$, and $g(T) = T^\gamma$, $T > 0$.

The proof of Theorem 1.8 can be found in Section 5. Under the assumptions of Theorem 1.8, the process $(X_t)_{t \in \mathbb{T}}$ is (T^β, T^γ) -dilatively stable for some unique $\beta, \gamma \in \mathbb{R}$, hence X could be called (β, γ) -dilatively stable, as well. However, for providing an easy link for comparison with Iglói's results [7], in Definition 1.2 we used the parametrization $(\alpha - \frac{\delta}{2}, \delta)$ instead of (β, γ) . Clearly, we have $\delta = \gamma$ and $\alpha = \beta + \frac{\delta}{2}$.

1.9 Remark. We give an example of a (f, g) -dilatively stable process such that f and g are not necessarily power functions. Namely, let $(W_t)_{t \in \mathbb{R}_+}$ be a standard Wiener process. Then the characteristic exponent of $(W_{t_1}, \dots, W_{t_k})$ takes the form

$$\psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = \frac{1}{2} \sum_{j, \ell=1}^k \min(t_j, t_\ell) \theta_j \theta_\ell$$

for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{R}_+$, and $\theta_1, \dots, \theta_k \in \mathbb{R}$. Hence $(W_t)_{t \in \mathbb{R}_+}$ is (f, g) -dilatively stable for any continuous functions $f, g : (0, \infty) \rightarrow (0, \infty)$ satisfying $g(T)f(T)^2 = T$, $T > 0$. Indeed, for all $T > 0$,

$$\begin{aligned} \psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k) &= \frac{1}{2} \sum_{j, \ell=1}^k \min(Tt_j, Tt_\ell) \theta_j \theta_\ell = T \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) \\ &= g(T)f(T)^2 \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = g(T) \frac{1}{2} \sum_{j, \ell=1}^k \min(t_j, t_\ell) f(T) \theta_j \cdot f(T) \theta_\ell \\ &= g(T) \psi_{t_1, \dots, t_k}(f(T)\theta_1, \dots, f(T)\theta_k). \end{aligned} \quad \square$$

Next, we recall some dilatively stable processes with stationary increments introduced by Iglói [7]: fractional Lévy process, and limits of integrated superposition of Ornstein–Uhlenbeck type processes, or of continuous-state continuous-time branching processes with immigration (such as the Cox–Ingersoll–Ross process). These processes have the same covariance function as a fractional Brownian motion for some parameter $H \in (\frac{1}{2}, 1)$. In fact, the non-Gaussian fractional Lévy process is $(H, 1)$ -dilatively stable, while the other processes are $(H, 2H - 2)$ -dilatively stable.

The remaining part of the paper is organized as follows. In Section 2 we give new examples for dilatively stable processes that are not self-similar, namely, we derive a sufficient condition under which a generalized fractional Lévy process is dilatively stable. As special cases, sub-fractional Lévy processes and log-fractional Lévy motions are dilatively stable. In Section 3 we consider limit processes appearing in certain aggregation models under an intermediate growth condition. In particular, the limit process of joint temporal and contemporaneous aggregation of independent copies of a stationary AR(1) process with random-coefficient, introduced in Pilipajsaute and Surgailis [15], is shown to be dilatively stable. Section 4 is devoted to the proof of Proposition 1.5, i.e., the connection between aggregate similarity and dilative stability. Finally, in Section 5 we prove Theorem 1.8.

2 Dilatively stable generalized fractional Lévy processes

First, we give a summary on two-sided Lévy processes based on Klüppelberg and Matsui [12, Section 2]. By a two-sided Lévy process $(L_t)_{t \in \mathbb{R}}$, we mean

$$L_t := L_t^{(1)} \mathbf{1}_{[0, \infty)}(t) - L_{(-t)-}^{(2)} \mathbf{1}_{(-\infty, 0)}(t), \quad t \in \mathbb{R},$$

where $L^{(1)}$ and $L^{(2)}$ are independent copies of a Lévy process. Moreover, we assume that L is centered (i.e., $\mathbb{E}(L_1) = 0$) without a Gaussian component, and its Lévy measure μ satisfies $\int_{|x|>1} x^2 \mu(dx) < \infty$, hence

$$\mathbb{E}(L_t^2) = |t| \mathbb{E}(L_1^2) = |t| \int_{\mathbb{R}} x^2 \mu(dx) < \infty, \quad t \in \mathbb{R}.$$

The above assumptions imply that the characteristic function of L_t , $t \in \mathbb{R}$, takes the form $\mathbb{E}(e^{i\theta L_t}) = e^{-|t|\varphi(\theta)}$, $t \in \mathbb{R}$, where

$$(2.1) \quad \varphi(\theta) := \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x) \mu(dx), \quad \theta \in \mathbb{R},$$

is the characteristic exponent of L_1 .

The following result is due to Marquardt [14, Proposition 2.1 and (2.13)].

2.1 Proposition. *Let $L = (L_t)_{t \in \mathbb{R}}$ be a two-sided Lévy process without a Gaussian component. Assume that $\mathbb{E}(L_1) = 0$ and $\mathbb{E}(L_1^2) < \infty$. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function such that $\mathbb{R} \ni u \mapsto f(t, u) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$. Then the integral $S_t := \int_{\mathbb{R}} f(t, u) L(du)$ exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ -sense for all $t \in \mathbb{R}$. Furthermore, for $s, t \in \mathbb{R}$ we obtain $\mathbb{E}(S_t) = 0$, the isometry*

$$\mathbb{E}(S_t^2) = \mathbb{E}(L_1^2) \int_{\mathbb{R}} f(t, u)^2 du, \quad t \in \mathbb{R},$$

and

$$\text{Cov}(S_s, S_t) = \mathbb{E}(L_1^2) \int_{\mathbb{R}} f(s, u) f(t, u) du, \quad s, t \in \mathbb{R}.$$

Moreover, the characteristic function of $(S_{t_1}, \dots, S_{t_k})$, $t_1 < t_2 < \dots < t_k$, $t_j \in \mathbb{R}$, $j \in \{1, \dots, k\}$, takes the form

$$(2.2) \quad \mathbb{E} \left(\exp \left\{ i \sum_{j=1}^k \theta_j S_{t_j} \right\} \right) = \exp \left\{ - \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k \theta_j f(t_j, u) \right) du \right\}$$

for $\theta_j \in \mathbb{R}$, $j \in \{1, \dots, k\}$, where φ is given by (2.1).

2.2 Definition. *A stochastic process $(S_t)_{t \in \mathbb{R}}$ given in Proposition 2.1 is called a generalized fractional Lévy process corresponding to the kernel function f .*

We note that Klüppelberg and Matsui [12, Definition 2.2] have already introduced generalized fractional Lévy processes with kernel function $f(t, u) := g(t - u) - g(-u)$, $t, u \in \mathbb{R}$, for some function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $g(t) = 0$ for $t < 0$ and $\int_{\mathbb{R}} (g(t - u) - g(-u))^2 du < \infty$.

Next we derive a sufficient condition under which a generalized fractional Lévy process is dilatively stable.

2.3 Proposition. *If the kernel function f given in Proposition 2.1 satisfies*

$$(2.3) \quad f(t, u) = T^{\alpha - \frac{\delta}{2}} f\left(\frac{t}{T}, \frac{u}{T^\delta}\right), \quad t, u \in \mathbb{R}, \quad T > 0,$$

then the generalized fractional Lévy process $(S_t)_{t \in \mathbb{R}}$ is (α, δ) -dilatively stable. If, in addition, $\mathbb{E}(L_1^2) > 0$ and $\int_{\mathbb{R}} f(1, u)^2 du > 0$, then the parameters of dilative stability α and δ are uniquely determined.

Proof. First observe that the process S is infinitely divisible, since by (2.2), for all $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, $t_1, \dots, t_k \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\mathbb{E} \left(\exp \left\{ i \sum_{j=1}^k \theta_j S_{t_j} \right\} \right) = \left(\exp \left\{ -\frac{1}{n} \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k \theta_j f(t_j, u) \right) du \right\} \right)^n,$$

where, again by (2.2),

$$\begin{aligned} \exp \left\{ -\frac{1}{n} \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k \theta_j f(t_j, u) \right) du \right\} &= \exp \left\{ -\int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k \theta_j f(t_j, nv) \right) dv \right\} \\ &= \mathbb{E} \left(\exp \left\{ i \sum_{j=1}^k \theta_j \tilde{S}_{t_j} \right\} \right) \end{aligned}$$

is the characteristic function of $(\tilde{S}_{t_1}, \dots, \tilde{S}_{t_k})$ with $\tilde{S}_t := \int_{\mathbb{R}} f(t, nv) L(dv)$, $t \in \mathbb{R}$.

Again by (2.2), the characteristic exponent of $(S_{t_1}, \dots, S_{t_k})$ has the form

$$\psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k \theta_j f(t_j, u) \right) du, \quad \theta_1, \dots, \theta_k \in \mathbb{R}.$$

Hence for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$ and $t_1, \dots, t_k \in \mathbb{R}$, by (2.3), we get

$$\begin{aligned} T^\delta \psi_{t_1, \dots, t_k}(T^{\alpha - \frac{\delta}{2}} \theta_1, \dots, T^{\alpha - \frac{\delta}{2}} \theta_k) &= T^\delta \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k T^{\alpha - \frac{\delta}{2}} \theta_j f(t_j, u) \right) du \\ &= \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k T^{\alpha - \frac{\delta}{2}} \theta_j f(Tt_j/T, v/T^\delta) \right) dv = \int_{\mathbb{R}} \varphi \left(\sum_{j=1}^k \theta_j f(Tt_j, v) \right) dv \\ &= \psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k), \end{aligned}$$

which yields that $(S_t)_{t \in \mathbb{R}}$ is (α, δ) -dilatively stable. If, in addition, $\mathbb{E}(L_1^2) > 0$ and $\int_{\mathbb{R}} f(1, u)^2 du > 0$, then, by Proposition 2.1, $\mathbb{E}(S_1^2) \in (0, \infty)$. Using also that S_1 is not Gaussian, part (iv) of Theorem 1.8 yields the uniqueness of α and δ . \square

Next we formulate three examples for generalized fractional Lévy processes that are dilatively stable, but not self-similar.

2.4 Example. (Sub-fractional Lévy process) Sub-fractional Brownian motions were introduced by Bojdecki et al. [4], see also Tudor [19, Section 1.3]. Let $(B_t^{(H)})_{t \in \mathbb{R}}$ be a fractional Brownian motion on \mathbb{R} with parameter $H \in (\frac{1}{2}, 1)$, i.e., $(B_t^{(H)})_{t \in \mathbb{R}}$ is a Gaussian process with zero mean having covariance function

$$\text{Cov}(B_t^{(H)}, B_s^{(H)}) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad t, s \in \mathbb{R}.$$

Let

$$S_t^{(H)} := \frac{1}{\sqrt{2}}(B_t^{(H)} + B_{-t}^{(H)}), \quad t \in \mathbb{R}_+.$$

Then $(S_t^{(H)})_{t \in \mathbb{R}_+}$ is a sub-fractional Brownian motion with parameter H , i.e., it is a Gaussian process with zero mean and with covariance function

$$\text{Cov}(S_t^{(H)}, S_s^{(H)}) = t^{2H} + s^{2H} - \frac{1}{2}((s+t)^{2H} + |t-s|^{2H}) \quad t, s \in \mathbb{R}_+,$$

see, e.g., Tudor [19, Proposition 1.17]. The process $S^{(H)}$ is H -self-similar, see, e.g., Tudor [19, Proposition 1.14]. Recall that

$$B_t^{(H)} = \frac{\sqrt{2H+1} \sin(\pi H)}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} ((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}) B(du), \quad t \in \mathbb{R},$$

where B is a Brownian motion on \mathbb{R} , see Samorodnitsky and Taqqu [16, Proposition 7.2.6] and Beran et al. [1, Section 1.3.5]. Hence we get

$$\begin{aligned} S_t^{(H)} = \frac{\sqrt{2H+1} \sin(\pi H)}{\sqrt{2} \Gamma(H + \frac{1}{2})} & \left[\int_{\mathbb{R}} ((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}) B(du) \right. \\ & \left. + \int_{\mathbb{R}} ((-t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}) B(du) \right], \quad t \in \mathbb{R}_+. \end{aligned}$$

Motivated by this moving average representation of $S^{(H)}$, we introduce so-called sub-fractional Lévy processes. Namely, for all $H \in (\frac{1}{2}, 1)$, let

$$\tilde{S}_t^{(H)} := \frac{1}{\sqrt{2}}(L_t^{(H)} + L_{-t}^{(H)}), \quad t \in \mathbb{R}_+,$$

where

$$L_t^{(H)} := \frac{\sqrt{2H+1} \sin(\pi H)}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} ((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}) L(du), \quad t \in \mathbb{R},$$

with a two-sided Lévy process $(L_t)_{t \in \mathbb{R}}$ without a Gaussian component satisfying $\mathbb{E}(L_1) = 0$ and $0 < \mathbb{E}(L_1^2) < \infty$. Note that $L_t^{(H)}$ is well-defined, since the kernel function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(t, u) := \begin{cases} (t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}, & \text{if } t \neq u \text{ and } u \neq 0, \\ 0, & \text{if } t = u \text{ or } u = 0, \end{cases}$$

satisfies the conditions of Proposition 2.1, see Beran et al. [1, Section 3.7.2.5].

We call the process $\tilde{S}^{(H)}$ a sub-fractional Lévy process with parameter H corresponding to the two-sided Lévy process L . The process $\tilde{S}^{(H)}$ is $(H, 1)$ -dilatively stable by Proposition 2.3, and the parameters are uniquely determined, since $\mathbb{E}(L_1^2) > 0$, and the kernel function f satisfies condition (2.3) with $\alpha = H$ and $\delta = 1$ together with $\int_{\mathbb{R}} f(1, u)^2 du > 0$. We note that the process $\tilde{S}^{(H)}$ has long memory increments, since $H \in (\frac{1}{2}, 1)$, see Klüppelberg and Matsui [12, Lemma 2.6]. \square

2.5 Example. (Log-fractional Lévy motion) Let $\alpha \in (1, 2)$ and M be a symmetric α -stable random measure with Lebesgue control measure. The process

$$X_t := \int_{\mathbb{R}} (\log(|t - u|) - \log(|u|)) M(du), \quad t \in \mathbb{R},$$

is called a symmetric log-fractional stable motion, see Samorodnitsky and Taqqu [16, Example 3.6.6]. For the definition of the integral with respect to M see [16, Section 3.4], and the fact that X_t , $t \in \mathbb{R}$, is well defined can be checked in the same way as in Samorodnitsky and Taqqu [16, Example 3.6.5]. Especially, $\int_{\mathbb{R}} (\log(|t - u|) - \log(|u|))^2 du < \infty$, $t \in \mathbb{R}$. The process X is $\frac{1}{\alpha}$ -self-similar with stationary increments. Motivated by this, let us introduce so-called log-fractional Lévy motions. Namely, we call the process

$$\tilde{X}_t := \int_{\mathbb{R}} (\log(|t - u|) - \log(|u|)) L(du), \quad t \in \mathbb{R},$$

a log-fractional Lévy motion, where $(L_t)_{t \in \mathbb{R}}$ is a two-sided Lévy process without a Gaussian component satisfying $\mathbb{E}(L_1) = 0$ and $0 < \mathbb{E}(L_1^2) < \infty$. Note that \tilde{X}_t is well-defined, since the kernel function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(t, u) := \begin{cases} \log(|t - u|) - \log(|u|), & \text{if } t \neq u \text{ and } u \neq 0, \\ 0, & \text{if } t = u \text{ or } u = 0, \end{cases}$$

satisfies the conditions of Proposition 2.1, see Samorodnitsky and Taqqu [16, Example 3.6.6].

The process $(\tilde{X}_t)_{t \in \mathbb{R}_+}$ is $(\frac{1}{2}, 1)$ -dilatively stable by Proposition 2.3, and the parameters are uniquely determined, since $\mathbb{E}(L_1^2) > 0$, and, for the kernel function f , condition (2.3) holds with $\alpha = \frac{1}{2}$ and $\delta = 1$ together with $\int_{\mathbb{R}} f(1, u)^2 du > 0$. \square

2.6 Example. Generalized sub-fractional Brownian motions were introduced by Sghir [17]. In one of the representation theorems of Sghir [17, Theorem 2.2], a self-similar process comes into play that can be generalized to present another example for a dilatively stable generalized fractional Lévy process as follows. For $K \in (0, 2)$, let

$$X_t^{(K)} := \int_{\mathbb{R}_+} (1 - e^{-ut}) u^{-\frac{K+1}{2}} B(du), \quad t \in \mathbb{R}_+,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion. The process $X^{(K)}$ is $\frac{K}{2}$ -self-similar. Let

$$\tilde{X}_t^{(K)} := \int_{\mathbb{R}_+} (1 - e^{-ut}) u^{-\frac{K+1}{2}} L(du), \quad t \in \mathbb{R}_+,$$

where $(L_t)_{t \in \mathbb{R}_+}$ is a two-sided Lévy process without a Gaussian component satisfying $\mathbb{E}(L_1) = 0$ and $0 < \mathbb{E}(L_1^2) < \infty$. Note that $\tilde{X}_t^{(K)}$ is well-defined, since the kernel function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(t, u) := \begin{cases} (1 - e^{-ut})u^{-\frac{K+1}{2}}, & \text{if } t > 0 \text{ and } u > 0, \\ 0, & \text{if } t \leq 0 \text{ or } u \leq 0, \end{cases}$$

satisfies the conditions of Proposition 2.1. Indeed, we must prove that

$$\int_{\mathbb{R}_+} (1 - e^{-ut})^2 u^{-(K+1)} du < \infty, \quad t > 0.$$

This integral may diverge at $u = 0$ or at $u = \infty$. As $u \rightarrow \infty$, the integrand behaves like $u^{-(K+1)}$, which is integrable, since $\int_1^\infty u^{-(K+1)} du < \infty$. As $u \downarrow 0$, the integrand behaves like $u^2 u^{-(K+1)} = u^{-K+1}$, which is integrable, since $\int_0^1 u^{-K+1} du < \infty$.

The process $\tilde{X}^{(K)}$ is $(\frac{K}{2}, -1)$ -dilatively stable by Proposition 2.3, and the parameters are uniquely determined, since $\mathbb{E}(L_1^2) > 0$, and one can check that condition (2.3) holds for the kernel function f with $\alpha = \frac{K}{2}$ and $\delta = -1$ together with $\int_{\mathbb{R}} f(1, u)^2 du > 0$. \square

In the next example we point out that under the conditions of Definition 1.2, the parameters α and δ are not unique in general.

2.7 Example. Let $\alpha \in (0, 2]$ and M be a symmetric α -stable random measure with Lebesgue control measure. For $H \in (0, 1)$, $H \neq \frac{1}{\alpha}$, a well-balanced linear fractional stable process $X = (X_t)_{t \in \mathbb{R}}$ is given by

$$X_t := \int_{\mathbb{R}} (|t - u|^{H-\frac{1}{\alpha}} - |u|^{H-\frac{1}{\alpha}}) M(du), \quad t \in \mathbb{R},$$

see, e.g., Samorodnitsky and Taqqu [16, Example 3.6.5]. On the one hand, it is known that X is H -self-similar and thus $(H, 0)$ -dilatively stable, since the process X is infinitely divisible. Further, the characteristic exponent of $(X_{t_1}, \dots, X_{t_k})$ can be written in the form

$$\psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = \int_{\mathbb{R}} \left| \sigma \sum_{j=1}^k \theta_j f(t_j, u) \right|^\alpha du$$

for $t_1 < \dots < t_k$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$ for some $\sigma > 0$, see, e.g., Samorodnitsky and Taqqu [16, Property 3.2.1], where the kernel function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, is given by

$$f(t, u) := \begin{cases} |t - u|^{H-1/\alpha} - |u|^{H-1/\alpha}, & \text{if } t \neq u \text{ and } u \neq 0, \\ 0, & \text{if } t = u \text{ or } u = 0. \end{cases}$$

Hence for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, and $t_1, \dots, t_k \in \mathbb{R}$, we get

$$\begin{aligned} \psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k) &= \int_{\mathbb{R}} \left| \sigma \sum_{j=1}^k \theta_j f(Tt_j, u) \right|^\alpha du = T \int_{\mathbb{R}} \left| \sigma \sum_{j=1}^k \theta_j f(Tt_j, Tv) \right|^\alpha dv \\ &= T \int_{\mathbb{R}} \left| \sigma \sum_{j=1}^k T^{H-1/\alpha} \theta_j f(t_j, v) \right|^\alpha dv = T \psi_{t_1, \dots, t_k}(T^{H-1/\alpha} \theta_1, \dots, T^{H-1/\alpha} \theta_k), \end{aligned}$$

showing that X is $(H - \frac{1}{\alpha} + \frac{1}{2}, 1)$ -dilatively stable. This example shows that the parametrization of dilatively stable processes is not unique in general. However, note also that the process M does not have finite second moments for $\alpha < 2$ and that for $\alpha = 2$ the linear fractional Brownian motion X is Gaussian. \square

3 Further examples from aggregation models

In this section we consider certain processes arising by aggregation of independent copies of random systems, appropriately rescaled in space and time. The behavior of such systems can vary significantly depending on the relative speed of aggregation with respect to the rescaling. We are particularly interested in intermediate regimes, where the speed of aggregation and rescaling is in a certain balance. In this way non self-similar, but dilatively stable limit processes may appear as follows, serving as further examples.

Our first example comes from a joint temporal and contemporaneous aggregation

$$(3.1) \quad S_{N,n}(t) = \sum_{i=1}^N \sum_{s=1}^{\lfloor nt \rfloor} X_i(s), \quad t \geq 0,$$

of N independent and identically distributed, stationary random coefficient AR(1) processes $(X_i(s))_{s \in \mathbb{R}_+}$, $i \in \{1, \dots, N\}$, with some given mixing density, recently investigated by Pilipajauskaitė and Surgailis [15]. Under an intermediate growth condition

$$(3.2) \quad \frac{N^{1/(1+\beta)}}{n} \rightarrow c \in (0, \infty)$$

as simultaneously $n, N \rightarrow \infty$, by Theorem 2.2 in [15] for any $\beta \in (-1, 1)$ the appropriately rescaled system (3.1) converges to a random process $(\sqrt{c}Z_\beta(t/c))_{t \in \mathbb{R}_+}$ in the sense of finite-dimensional distributions. Here the process Z_β is infinitely divisible by Proposition 3.1 in [15] and determined by the characteristic exponent of $(Z_\beta(t_1), \dots, Z_\beta(t_k))$ which takes the form

$$\psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = -C \int_0^\infty \left(\exp \left\{ -\frac{1}{2} \int_{\mathbb{R}} \left(\sum_{j=1}^k \theta_j (f(x, t_j - s) - f(x, -s)) \right)^2 ds \right\} - 1 \right) x^\beta dx,$$

for all $t_j \in \mathbb{R}_+$, $\theta_j \in \mathbb{R}$, $j = 1, \dots, k$, $k \in \mathbb{N}$, and some constant $C > 0$, where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$(3.3) \quad f(x, t) := \begin{cases} \frac{1 - e^{-xt}}{x} & \text{if } x > 0 \text{ and } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As remarked on page 1022 of [15] the process Z_β is not self-similar and not stable.

3.1 Proposition. *For any $\beta \in (-1, 1)$, the process Z_β is $(1 - \frac{\beta}{2}, -\beta - 1)$ -dilatively stable.*

Proof. By Proposition 3.1 in Pilipajsaute and Surgailis [15], Z_β is infinitely divisible. Note that the function f in (3.3) fulfills $f(x, Tt) = T \cdot f(Tx, t)$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $T > 0$. Hence, for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$ and $t_1, \dots, t_k \in \mathbb{R}_+$, by change of variables $s = Tu$ and then $Tx = y$, we get

$$\begin{aligned}
& \psi_{Tt_1, \dots, Tt_k}(\theta_1, \dots, \theta_k) \\
&= -C \int_0^\infty \left(\exp \left\{ -\frac{1}{2} \int_{\mathbb{R}} \left(\sum_{j=1}^k \theta_j (f(x, Tt_j - s) - f(x, -s)) \right)^2 ds \right\} - 1 \right) x^\beta dx \\
&= -C \int_0^\infty \left(\exp \left\{ -\frac{1}{2} \int_{\mathbb{R}} \left(\sum_{j=1}^k T^{3/2} \theta_j (f(Tx, t_j - u) - f(Tx, -u)) \right)^2 du \right\} - 1 \right) x^\beta dx \\
&= -T^{-\beta-1} C \int_0^\infty \left(\exp \left\{ -\frac{1}{2} \int_{\mathbb{R}} \left(\sum_{j=1}^k T^{3/2} \theta_j (f(y, t_j - u) - f(y, -u)) \right)^2 du \right\} - 1 \right) y^\beta dy \\
&= T^{-\beta-1} \psi_{t_1, \dots, t_k}(T^{3/2} \theta_1, \dots, T^{3/2} \theta_k).
\end{aligned}$$

By definition of dilative stability, this yields the assertion. \square

Note that, by Proposition 3.1 in [15], for $\beta \in (0, 1)$ the process Z_β is centered with $\mathbb{E}(Z_\beta(1)^2) \in (0, \infty)$, and for $\beta \in (-\frac{1}{2}, 1)$ a.s. has continuous trajectories. Since $Z_\beta(1)$ is not Gaussian, part (iv) of Theorem 1.8 shows that at least for $\beta \in (0, 1)$ the parameters of dilative stability for Z_β are unique.

An alternative way to verify dilative stability of Z_β is by means of aggregate similarity introduced in Definition 1.4. As remarked on page 1023 in [15], the process $(Z_\beta(t^{2/3}))_{t \geq 0}$ is $(\frac{3}{2(1+\beta)}, \frac{3}{2(1+\beta)})$ -aggregate similar, where $\beta \in (-1, 1)$. Hence, by an easy calculation, Z_β is $(\frac{3}{2(1+\beta)}, \frac{1}{1+\beta})$ -aggregate similar. Then, since Z_β is infinitely divisible (see Proposition 3.1 in [15]), part (ii) of Proposition 1.5 shows that Z_β is $(1 - \frac{\beta}{2}, -\beta - 1)$ -dilatively stable at least for the reduced parameter set $\beta \in (-\frac{1}{2}, 1)$, where weak right-continuity is justified by a.s. continuity of the sample paths.

Note that, by Theorems 2.1 and 2.2 in [15], the limit process of the appropriately rescaled system (3.1) under slow and fast growth condition, i.e., $c = 0$ or $c = \infty$ in (3.2), or under iterated aggregation $\lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty}$ or $\lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty}$ is either fractional Brownian motion, or a linear time multiple of a stable random variable, or a variance mixture of Brownian motion with a stable mixing variable. Since all of these processes are self-similar, they are naturally dilatively stable as well. All in all, each of the limit processes in Theorems 2.1 and 2.2 in [15] is dilatively stable.

Kaj's motivation to introduce the notion of aggregate similarity in [9] comes from an aggregation model having its origin in the study of total workload in teletraffic models. The survey in [9] shows that in several different models (superposition of renewal counting processes, sums of inverse Lévy subordinators, infinite source Poisson models, self-similar rate models, inference

model for wireless communication) under an intermediate growth condition the same scaling limit process Y_γ arises, depending on a stability parameter $\gamma \in (1, 2)$; cf. also [5, 6, 10, 11]. By Theorem 2 in [6], this process is particularly not self-similar and not stable but Y_γ has a.s. continuous trajectories. Since Y_γ is aggregate similar with rigidity index $\varrho = (\gamma - 1)^{-1}$ by Section 3.6 in [9] and is infinitely divisible by Section 3.4 in [5], it is also $(\frac{3-\gamma}{2}, 1 - \gamma)$ -dilatively stable by part (ii) of Proposition 1.5. Further, Gaigalas [5, Sections 3.1, 3.2 and 4] has shown that the process Y_γ builds a certain bridge between a γ -stable Lévy process and fractional Brownian motion with Hurst index $H = \frac{3-\gamma}{2}$ appearing as scaling limits in the above mentioned models under slow, respectively fast growth condition. Note that the γ -stable Lévy process is also $(\frac{3-\gamma}{2}, 1 - \gamma)$ -dilatively stable (see the paragraph before Remark 1.3) and the $\frac{3-\gamma}{2}$ -selfsimilar fractional Brownian motion is also $(\frac{3-\gamma}{2}, 0)$ -dilatively stable. Hence all the possible limit processes are (α, δ) -dilatively stable with the same parameter $\alpha = \frac{3-\gamma}{2}$ giving a justification for Iglói's parametrization (see the comment after Theorem 1.8).

Since, by Theorem 4 in [9], the process Y_γ has an integral representation with respect to a compensated Poisson random measure for which a natural extension to a multivariate spatial model has been given by Biermé et al. [2], it seems to be possible to generalize the notion of dilative stability to higher dimensions. The characteristic function of the random field in question can be found on page 1139 in [2] and suggests that the scaling relation of dilative stability is fulfilled. However, we renounce to consider this question in full detail, since for random fields even a more general scaling by linear operators as in [3] might be introduced, giving rise for future research.

4 Proof of Proposition 1.5

First, note that in terms of characteristic exponents, for an infinitely divisible process $(X_t)_{t \in \mathbb{T}}$, the scaling relation (1.6) is equivalent to

$$(4.1) \quad m \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = \psi_{m^{-\varrho_2} t_1, \dots, m^{-\varrho_2} t_k}(m^{\varrho_1} \theta_1, \dots, m^{\varrho_1} \theta_k)$$

for every $m \in \mathbb{N}$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, and $t_1, \dots, t_k \in \mathbb{T}$.

Suppose that $(X_t)_{t \in \mathbb{T}}$ is a (α, δ) -dilatively stable process for some $(\alpha, \delta) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. By (1.3),

$$\psi_{T t_1, \dots, T t_k}(\theta_1, \dots, \theta_k) = T^\delta \psi_{t_1, \dots, t_k}(T^{\alpha - \frac{\delta}{2}} \theta_1, \dots, T^{\alpha - \frac{\delta}{2}} \theta_k)$$

for all $T > 0$, $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in \mathbb{R}$, and $t_1, \dots, t_k \in \mathbb{T}$. By choosing $T = m^{1/\delta}$, $m \in \mathbb{N}$, we have

$$\psi_{m^{1/\delta} t_1, \dots, m^{1/\delta} t_k}(\theta_1, \dots, \theta_k) = m \psi_{t_1, \dots, t_k}(m^{\frac{\alpha}{\delta} - \frac{1}{2}} \theta_1, \dots, m^{\frac{\alpha}{\delta} - \frac{1}{2}} \theta_k).$$

Replacing θ_i by $m^{\frac{1}{2} - \frac{\alpha}{\delta}} \theta_i$, $i \in \{1, \dots, k\}$, we have (4.1) with $\varrho_1 = \frac{1}{2} - \frac{\alpha}{\delta}$ and $\varrho_2 = -\frac{1}{\delta}$, concluding part (i).

Let us now suppose that $(X_t)_{t \in \mathbb{T}}$ is an infinitely divisible, weakly right-continuous (ϱ_1, ϱ_2) -aggregate similar process with some $(\varrho_1, \varrho_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. Then, by letting $s_i := m^{-\varrho_2} t_i$, $\tilde{\theta}_i := m^{\varrho_1} \theta_i$, $i \in \{1, \dots, k\}$, we have that (4.1) is equivalent to

$$(4.2) \quad \psi_{m^{\varrho_2} s_1, \dots, m^{\varrho_2} s_k}(m^{-\varrho_1} \tilde{\theta}_1, \dots, m^{-\varrho_1} \tilde{\theta}_k) = m^{-1} \psi_{s_1, \dots, s_k}(\tilde{\theta}_1, \dots, \tilde{\theta}_k)$$

for all $m \in \mathbb{N}$, $k \in \mathbb{N}$, $s_1, \dots, s_k \in \mathbb{T}$ and $\tilde{\theta}_1, \dots, \tilde{\theta}_k \in \mathbb{R}$. Now let $z = \frac{m}{n}$ be an arbitrary positive rational number. By (4.1) and (4.2), we have

$$(4.3) \quad \begin{aligned} z \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) &= n^{-1} \psi_{m^{-\varrho_2} t_1, \dots, m^{-\varrho_2} t_k}(m^{\varrho_1} \theta_1, \dots, m^{\varrho_1} \theta_k) \\ &= n^{-1} \psi_{n^{-\varrho_2} z^{-\varrho_2} t_1, \dots, n^{-\varrho_2} z^{-\varrho_2} t_k}(n^{\varrho_1} z^{\varrho_1} \theta_1, \dots, n^{\varrho_1} z^{\varrho_1} \theta_k) \\ &= \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z^{\varrho_1} \theta_1, \dots, z^{\varrho_1} \theta_k) \end{aligned}$$

for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{T}$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$.

Finally, if $z > 0$ is arbitrary, then choose a sequence $(z_n)_{n \in \mathbb{N}}$ of positive rational numbers such that $z_n \downarrow z$ as $n \rightarrow \infty$. By (4.3), we get

$$\begin{aligned} z \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) &= \lim_{n \rightarrow \infty} z_n \psi_{t_1, \dots, t_k}(\theta_1, \dots, \theta_k) = \lim_{n \rightarrow \infty} \psi_{z_n^{-\varrho_2} t_1, \dots, z_n^{-\varrho_2} t_k}(z_n^{\varrho_1} \theta_1, \dots, z_n^{\varrho_1} \theta_k) \\ &= \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z^{\varrho_1} \theta_1, \dots, z^{\varrho_1} \theta_k), \end{aligned}$$

where the last equality can be checked as follows. By Lévy's continuity theorem, weak right-continuity of the finite-dimensional distributions of $(X_t)_{t \in \mathbb{T}}$ yields $\mathbb{E}(\exp\{i \sum_{j=1}^k \theta_j X_{z_n^{-\varrho_2} t_j}\}) \rightarrow \mathbb{E}(\exp\{i \sum_{j=1}^k \theta_j X_{z^{-\varrho_2} t_j}\})$ as $n \rightarrow \infty$ for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{T}$, uniformly on compact subsets of $(\theta_1, \dots, \theta_k)^\top \in \mathbb{R}^k$. Further, by Lemma 3.2.11 in Stroock [18], we conclude continuity of the characteristic exponent ψ_{t_1, \dots, t_k} for all $k \in \mathbb{N}$ and $t_1, \dots, t_k \in \mathbb{T}$, and $\psi_{z_n^{-\varrho_2} t_1, \dots, z_n^{-\varrho_2} t_k}(\theta_1, \dots, \theta_k) \rightarrow \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(\theta_1, \dots, \theta_k)$ as $n \rightarrow \infty$ uniformly on compacts for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{T}$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$. Consequently, denoting the closed ball around $(z^{\varrho_1} \theta_1, \dots, z^{\varrho_1} \theta_k)$ with radius 1 by $B_{\theta_1, \dots, \theta_k}^{z, \varrho_1}$, we have, for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} &|\psi_{z_n^{-\varrho_2} t_1, \dots, z_n^{-\varrho_2} t_k}(z_n^{\varrho_1} \theta_1, \dots, z_n^{\varrho_1} \theta_k) - \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z^{\varrho_1} \theta_1, \dots, z^{\varrho_1} \theta_k)| \\ &\leq |\psi_{z_n^{-\varrho_2} t_1, \dots, z_n^{-\varrho_2} t_k}(z_n^{\varrho_1} \theta_1, \dots, z_n^{\varrho_1} \theta_k) - \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z_n^{\varrho_1} \theta_1, \dots, z_n^{\varrho_1} \theta_k)| \\ &\quad + |\psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z_n^{\varrho_1} \theta_1, \dots, z_n^{\varrho_1} \theta_k) - \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z^{\varrho_1} \theta_1, \dots, z^{\varrho_1} \theta_k)| \\ &\leq \sup_{x \in B_{\theta_1, \dots, \theta_k}^{z, \varrho_1}} |\psi_{z_n^{-\varrho_2} t_1, \dots, z_n^{-\varrho_2} t_k}(x) - \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(x)| \\ &\quad + |\psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z_n^{\varrho_1} \theta_1, \dots, z_n^{\varrho_1} \theta_k) - \psi_{z^{-\varrho_2} t_1, \dots, z^{-\varrho_2} t_k}(z^{\varrho_1} \theta_1, \dots, z^{\varrho_1} \theta_k)| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Letting $T = z^{-\varrho_2}$ and $\tilde{\theta}_i = z^{\varrho_1} \theta_i$, $i \in \{1, \dots, k\}$, we have

$$\psi_{T t_1, \dots, T t_k}(\tilde{\theta}_1, \dots, \tilde{\theta}_k) = T^{-\frac{1}{\varrho_2}} \psi_{t_1, \dots, t_k}(T^{\frac{\varrho_1}{\varrho_2}} \tilde{\theta}_1, \dots, T^{\frac{\varrho_1}{\varrho_2}} \tilde{\theta}_k)$$

for all $k \in \mathbb{N}$, $T > 0$, $t_1, \dots, t_k \in \mathbb{T}$ and $\tilde{\theta}_1, \dots, \tilde{\theta}_k \in \mathbb{R}$, which coincides with the definition of $(\frac{\varrho_1}{\varrho_2} - \frac{1}{2\varrho_2}, -\frac{1}{\varrho_2})$ -dilative stability concluding part (ii). \square

5 Proof of Theorem 1.8

Applying (1.7) with $k = 1$ and $t_1 = 1$, we get

$$(5.1) \quad \psi_T(\theta) = g(T)\psi_1(f(T)\theta), \quad \theta \in \mathbb{R}, \quad T \in (0, \infty).$$

Replacing T by ts with $s, t \in (0, \infty)$, we have

$$\psi_{ts}(\theta) = g(ts)\psi_1(f(ts)\theta), \quad \theta \in \mathbb{R}, \quad s, t \in (0, \infty).$$

Using (1.7) with $k = 1$, $T = t$ and $t_1 = s$, and applying (5.1), we get

$$\psi_{ts}(\theta) = g(t)\psi_s(f(t)\theta) = g(t)g(s)\psi_1(f(t)f(s)\theta), \quad \theta \in \mathbb{R}, \quad s, t \in (0, \infty).$$

The uniqueness of the characteristic exponent of the infinitely divisible random variable X_{ts} implies that

$$(5.2) \quad g(st)\psi_1(f(st)\theta) = g(s)g(t)\psi_1(f(s)f(t)\theta), \quad \theta \in \mathbb{R}, \quad s, t \in (0, \infty).$$

If there exists a $\beta \in \mathbb{R}$ such that $f(T) = T^\beta$, $T \in (0, \infty)$, then (5.2) takes the form

$$(5.3) \quad g(st)\psi_1((st)^\beta\theta) = g(s)g(t)\psi_1((st)^\beta\theta), \quad \theta \in \mathbb{R}, \quad s, t \in (0, \infty).$$

If X is not the identically zero process, then ψ_1 is not the identically zero function, hence there exists $\theta_0 \in \mathbb{R}$ such that $\psi_1(\theta_0) \neq 0$. Substituting $\theta = \theta_0/(st)^\beta$ into (5.3) and dividing both sides by $\psi_1(\theta_0)$, we obtain $g(st) = g(s)g(t)$ for all $s, t \in (0, \infty)$. Since g is supposed to be continuous, it is known that there exists some $\gamma \in \mathbb{R}$ such that $g(t) = t^\gamma$, $t \in (0, \infty)$. Substituting $\theta = \theta_0/T^\beta$ into (5.1) and dividing both sides by $\psi_1(\theta_0)$, we get $g(T) = \psi_T(\theta_0/T^\beta)/\psi_1(\theta_0)$ for all $T \in (0, \infty)$, hence we obtain the uniqueness of γ , and we conclude statement (i).

Recall that there exist some $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_{++}$ and a measure μ on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R} \setminus \{0\}} \min(1, x^2) \mu(dx) < \infty$ (called Lévy measure) such that

$$(5.4) \quad \psi_1(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{i\theta x} + \frac{i\theta x}{1+x^2}\right) \mu(dx), \quad \theta \in \mathbb{R}.$$

Hence we get

$$\begin{aligned} g(st)\psi_1(f(st)\theta) &= ig(st)f(st)a\theta + \frac{1}{2}g(st)f(st)^2\sigma^2\theta^2 \\ &\quad + g(st) \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{if(st)\theta x} + \frac{if(st)\theta x}{1+x^2}\right) \mu(dx) \\ &= i\tilde{a}\theta + \frac{1}{2}\tilde{\sigma}^2\theta^2 + \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{i\theta x} + \frac{i\theta x}{1+x^2}\right) \tilde{\mu}(dx), \quad \theta \in \mathbb{R}, \quad s, t \in (0, \infty), \end{aligned}$$

where $\tilde{\sigma}^2 := g(st)f(st)^2\sigma^2$, $\tilde{\mu}(dx) := g(st)\mu\left(\frac{dx}{f(st)}\right)$ is a Lévy measure, and

$$\begin{aligned} i\tilde{a}\theta &:= ig(st)f(st)a\theta + g(st) \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{if(st)\theta x} + \frac{if(st)\theta x}{1+x^2}\right) \mu(dx) \\ &\quad - \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + \frac{i\theta x}{1+x^2}\right) \tilde{\mu}(dx) \\ &= i\theta g(st) \left(f(st)a + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{y}{1+(y/f(st))^2} - \frac{y}{1+y^2}\right) \mu\left(\frac{dy}{f(st)}\right)\right) \\ &= i\theta g(st)f(st) \left(a + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+x^2} - \frac{x}{1+f(st)^2x^2}\right) \mu(dx)\right). \end{aligned}$$

Similarly,

$$g(s)g(t)\psi_1(f(s)f(t)\theta) = i\hat{a}\theta + \frac{1}{2}\hat{\sigma}^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\theta x} + \frac{i\theta x}{1+x^2}\right) \hat{\mu}(dx),$$

for all $\theta \in \mathbb{R}$ and $s, t \in (0, \infty)$, where $\hat{\sigma}^2 := g(s)g(t)f(s)^2f(t)^2\sigma^2$, $\hat{\mu}(dx) = g(s)g(t)\mu\left(\frac{dx}{f(s)f(t)}\right)$ is a Lévy measure, and

$$\hat{a} := g(s)g(t)f(s)f(t) \left(a + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{x}{1+x^2} - \frac{x}{1+f(s)^2f(t)^2x^2}\right) \mu(dx)\right).$$

Since the triplet corresponding to the infinitely divisible random variable X_{st} is uniquely determined provided that the truncation function is fixed (see, e.g., Stroock [18, Theorem 3.2.20]), we have $\tilde{a} = \hat{a}$, $\tilde{\sigma}^2 = \hat{\sigma}^2$ and $\tilde{\mu} = \hat{\mu}$.

Now we suppose that X_1 is not Gaussian, but has a Gaussian component in its Lévy–Khintchine representation, i.e., $\sigma \neq 0$ and $\mu \neq 0$. Then $\tilde{\sigma}^2 = \hat{\sigma}^2$ yields that $h(st) = h(s)h(t)$, $s, t \in (0, \infty)$, for the function $h : (0, \infty) \rightarrow (0, \infty)$, $h(t) := g(t)f(t)^2$, $t \in (0, \infty)$. Since f and g are supposed to be continuous, it is known that there exists some $\kappa \in \mathbb{R}$ such that $g(t)f(t)^2 = t^\kappa$, $t \in (0, \infty)$. Then $\tilde{\mu} = \hat{\mu}$ implies $g(st)\mu\left(\frac{dx}{f(st)}\right) = g(s)g(t)\mu\left(\frac{dx}{f(s)f(t)}\right)$ for all $s, t \in (0, \infty)$. Since μ is a Lévy measure, we have $\mu((y, \infty)) < \infty$ for all $y \in (0, \infty)$, hence

$$g(st)\mu\left(\left(\frac{y}{f(st)}, \infty\right)\right) = g(s)g(t)\mu\left(\left(\frac{y}{f(s)f(t)}, \infty\right)\right), \quad y, s, t \in (0, \infty).$$

Consequently, $\mu((z, \infty)) = a\mu((bz, \infty))$ for all $z \in (0, \infty)$, where $a := g(s)g(t)/g(st) \in (0, \infty)$ and $b := f(st)/(f(s)f(t)) \in (0, \infty)$. We have $a = b^2$, since we have already proven that $g(t)f(t)^2 = t^\kappa$ for all $t \in (0, \infty)$. The aim of the following discussion is to show that $a \in (0, 1)$ would lead to a contradiction. If so, then clearly $b \in (0, 1)$. Iterating $\mu(dz) = a\mu(bdz)$, we obtain $\mu(dz) = a^n\mu(b^n dz)$ for all $n \in \mathbb{N}$. Since $\mu \neq 0$, without loss of generality, we may suppose that $\mu((0, \infty)) > 0$. Then there exists $z_0 \in (0, \infty)$ such that $\mu((z_0, \infty)) > 0$. Then $\mu((z_0, \infty)) = a^n\mu((b^n z_0, \infty))$ for all $n \in \mathbb{N}$. Since $\mu((z_0, \infty)) > 0$, $a^n \rightarrow 0$ and $\mu((b^n z_0, \infty)) \rightarrow \mu((0, \infty))$ as $n \rightarrow \infty$, we conclude $\mu((0, \infty)) = \infty$. Consequently, using

that μ is a Lévy measure, $\int_{(0,1]} z^2 \mu(dz) \in (0, \infty)$. Applying $\mu(dz) = a^n \mu(b^n dz)$, $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \int_{(0,1]} z^2 \mu(dz) &= \int_{(0,1]} z^2 a^n \mu(b^n dz) = a^n \int_{(0,b^n]} \left(\frac{y}{b^n}\right)^2 \mu(dy) \\ &= \left(\frac{a}{b^2}\right)^n \int_{(0,b^n]} y^2 \mu(dy) = \int_{(0,b^n]} y^2 \mu(dy), \end{aligned}$$

and $\lim_{n \rightarrow \infty} \int_{(0,b^n]} y^2 \mu(dy) = 0$ leads to a contradiction. In a similar way, $a \in (1, \infty)$ would also lead to a contradiction, since then we would have $\mu((z, \infty)) = a^{-1} \mu((b^{-1}z, \infty))$ for all $z \in (0, \infty)$, where $a^{-1} \in (0, 1)$. Summarizing, the only possibility is $a = 1$, which implies $b = 1$, and hence $f(st) = f(s)f(t)$, $s, t \in (0, \infty)$, and $g(st) = g(s)g(t)$, $s, t \in (0, \infty)$. Since f and g are supposed to be continuous, it is known that there exist some $\beta, \gamma \in \mathbb{R}$ such that $f(t) = t^\beta$, $t \in (0, \infty)$, and $g(t) = t^\gamma$, $t \in (0, \infty)$. Now, we show that β and γ are uniquely determined. Using (5.1) and (5.4), we obtain

$$\begin{aligned} \psi_T(\theta) &= iaT^{\beta+\gamma}\theta + \frac{1}{2}\sigma^2 T^{2\beta+\gamma}\theta^2 + T^\gamma \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{iT^\beta \theta x} + \frac{iT^\beta \theta x}{1+x^2}\right) \mu(dx) \\ &= i \left(a + \int_{\mathbb{R} \setminus \{0\}} \left(\frac{x}{1+x^2} - \frac{x}{1+T^{2\beta}x^2}\right) \mu(dx)\right) T^{\beta+\gamma}\theta + \frac{1}{2}\sigma^2 T^{2\beta+\gamma}\theta^2 \\ &\quad + \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{i\theta x} + \frac{i\theta x}{1+x^2}\right) T^\gamma \mu\left(\frac{dx}{T^\beta}\right) \end{aligned}$$

for all $\theta \in \mathbb{R}$ and $T \in (0, \infty)$. If there exist some $\tilde{\beta}, \tilde{\gamma} \in \mathbb{R}$ such that the process $(X_t)_{t \in \mathbb{T}}$ is (\tilde{f}, \tilde{g}) -dilatively stable with $\tilde{f}(t) = t^{\tilde{\beta}}$, $t \in (0, \infty)$, and $g(t) = t^{\tilde{\gamma}}$, $t \in (0, \infty)$, then, by the same reasoning,

$$\begin{aligned} \psi_T(\theta) &= i \left(a + \int_{\mathbb{R} \setminus \{0\}} \left(\frac{x}{1+x^2} - \frac{x}{1+T^{2\tilde{\beta}}x^2}\right) \mu(dx)\right) T^{\tilde{\beta}+\tilde{\gamma}}\theta + \frac{1}{2}\sigma^2 T^{2\tilde{\beta}+\tilde{\gamma}}\theta^2 \\ &\quad + \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{i\theta x} + \frac{i\theta x}{1+x^2}\right) T^{\tilde{\gamma}} \mu\left(\frac{dx}{T^{\tilde{\beta}}}\right) \end{aligned}$$

for all $\theta \in \mathbb{R}$ and $T \in (0, \infty)$. Since the triplet corresponding to the infinitely divisible random variable X_T is uniquely determined provided that the truncation function is fixed (see, e.g., Stroock [18, Theorem 3.2.20]), we have $\sigma^2 T^{2\beta+\gamma} = \sigma^2 T^{2\tilde{\beta}+\tilde{\gamma}}$, $T \in (0, \infty)$, and $T^\gamma \mu\left(\frac{dx}{T^\beta}\right) = T^{\tilde{\gamma}} \mu\left(\frac{dx}{T^{\tilde{\beta}}}\right)$, $T \in (0, \infty)$. Thus $\sigma \neq 0$ implies $2\beta + \gamma = 2\tilde{\beta} + \tilde{\gamma}$, and hence $\tilde{\gamma} - \gamma = 2(\beta - \tilde{\beta})$, which yields $\mu(dy) = T^{\tilde{\gamma}-\gamma} \mu(T^{\beta-\tilde{\beta}} dy) = T^{2(\beta-\tilde{\beta})} \mu(T^{\beta-\tilde{\beta}} dy)$, $T \in (0, \infty)$. The aim of the following discussion is to show that $\beta < \tilde{\beta}$ would lead us to a contradiction. Since $\mu \neq 0$, without loss of generality, we may suppose that $\mu((0, \infty)) > 0$. Then there exists $y_0 \in (0, \infty)$ such that $\mu((y_0, \infty)) > 0$. Then $\mu((y_0, \infty)) = T^{2(\beta-\tilde{\beta})} \mu((T^{\beta-\tilde{\beta}} y_0, \infty))$ for all $T \in (0, \infty)$. Since $\mu((y_0, \infty)) > 0$, $T^{2(\beta-\tilde{\beta})} \rightarrow 0$ and $\mu((T^{\beta-\tilde{\beta}} y_0, \infty)) \rightarrow \mu((0, \infty))$ as

$T \rightarrow \infty$, we conclude $\mu((0, \infty)) = \infty$. Consequently, using again that μ is a Lévy measure, $\int_{(0,1]} y^2 \mu(dy) \in (0, \infty)$. Applying $\mu(dy) = T^{2(\beta-\tilde{\beta})} \mu(T^{\beta-\tilde{\beta}} dy)$, $T \in (0, \infty)$, we obtain

$$\begin{aligned} \int_{(0,1]} y^2 \mu(dy) &= \int_{(0,1]} y^2 T^{2(\beta-\tilde{\beta})} \mu(T^{\beta-\tilde{\beta}} dy) = T^{2(\beta-\tilde{\beta})} \int_{(0, T^{\beta-\tilde{\beta}}]} \left(\frac{z}{T^{\beta-\tilde{\beta}}} \right)^2 \mu(dz) \\ &= \int_{(0, T^{\beta-\tilde{\beta}}]} z^2 \mu(dz), \end{aligned}$$

and $\lim_{T \rightarrow \infty} \int_{(0, T^{\beta-\tilde{\beta}}]} y^2 \mu(dy) = 0$ leads us to a contradiction. In a similar way, $\beta > \tilde{\beta}$ would also lead us to a contradiction, since then we would have $\mu((z, \infty)) = T^{2(\tilde{\beta}-\beta)} \mu((T^{\tilde{\beta}-\beta} z, \infty))$ for all $z \in (0, \infty)$. Summarizing, the only possibility is $\beta = \tilde{\beta}$, which implies $\gamma = \tilde{\gamma}$, and we conclude statement (ii).

Next we suppose that $\mathbb{E}(X_1^2) < \infty$, $\mathbb{E}(X_1) \neq 0$ and $\text{Var}(X_1) > 0$. Then ψ_1 is twice differentiable and $\mathbb{E}(X_T) = i\psi'_T(0)$, $T \in (0, \infty)$, and $\text{Var}(X_T) = \psi''_T(0)$, $T \in (0, \infty)$. Taking the derivative of (5.2) and putting $\theta = 0$, we obtain

$$g(st)f(st)\psi'_1(0) = g(s)g(t)f(s)f(t)\psi'_1(0), \quad s, t \in (0, \infty).$$

Since $\psi'_1(0) = -i\mathbb{E}(X_1) \neq 0$, we conclude that there exists some $\lambda \in \mathbb{R}$ such that $g(t)f(t) = t^\lambda$, $t \in (0, \infty)$. Taking the second derivative of (5.2) and putting $\theta = 0$, we get

$$g(st)f(st)^2\psi''_1(0) = g(s)g(t)f(s)^2f(t)^2\psi''_1(0), \quad s, t \in (0, \infty).$$

Since $\psi''_1(0) = \text{Var}(X_1) \neq 0$, we obtain that there exists some $\kappa \in \mathbb{R}$ such that $g(t)f(t)^2 = t^\kappa$, $t \in (0, \infty)$. Since f and g are positive, it yields that $f(t) = t^{\kappa-\lambda}$, $t \in (0, \infty)$, and $g(t) = t^{2\lambda-\kappa}$, $t \in (0, \infty)$. Choosing $\beta := \kappa - \lambda$ and $\gamma := 2\lambda - \kappa$, we have $f(t) = t^\beta$, $t \in (0, \infty)$, and $g(t) = t^\gamma$, $t \in (0, \infty)$. Next we show that β and γ are uniquely determined. Using (5.1), we get

$$(5.5) \quad \psi_T(\theta) = T^\gamma \psi_1(T^\beta \theta), \quad \theta \in \mathbb{R}, \quad T \in (0, \infty).$$

Taking the derivative of (5.5) and putting $\theta = 0$, we obtain $\psi'_T(0) = T^{\beta+\gamma} \psi'_1(0)$, $T \in (0, \infty)$. Since $\psi'_1(0) = -i\mathbb{E}(X_1) \neq 0$, we conclude that $\beta + \gamma$ is uniquely determined. Taking the second derivative of (5.5) and putting $\theta = 0$, we obtain $\psi''_T(0) = T^{2\beta+\gamma} \psi''_1(0)$, $T \in (0, \infty)$. Since $\psi''_1(0) = \text{Var}(X_1) \neq 0$, we have that $2\beta + \gamma$ is also uniquely determined, thus we get the uniqueness of β and γ , and we conclude statement (iii).

Finally, we suppose that X_1 is not Gaussian, $\mathbb{E}(X_1^2) < \infty$ and $\text{Var}(X_1) > 0$. As in the proof of (iii), there exists some $\kappa \in \mathbb{R}$ such that $g(t)f(t)^2 = t^\kappa$, $t \in (0, \infty)$; further, provided that there exist some $\beta, \gamma \in \mathbb{R}$ such that $f(t) = t^\beta$, $t \in (0, \infty)$, and $g(t) = t^\gamma$, $t \in (0, \infty)$, then $2\beta + \gamma$ is uniquely determined. The proof of statement (iv) can be finished as in the case of (ii). \square

5.1 Remark. (i) The proof of part (iii) of Theorem 1.8 uses the same ideas as that of Theorem 2.2.1 in Iglói [7], however, instead of cumulants we work with moments, and we have much weaker (moment) assumptions.

(ii) Similarly to the proof of part (ii) of Theorem 1.8, one can check that if $(X_t)_{t \in \mathbb{T}}$ is an (α, δ) -dilatively stable process for some $\alpha, \delta \in \mathbb{R}$ such that $\mathbb{E}(X_1^2) < \infty$, then

$$\mathbb{E}(X_{Tt}) = T^{\alpha + \frac{\delta}{2}} \mathbb{E}(X_t) \quad \text{and} \quad \text{Var}(X_{Tt}) = T^{2\alpha} \text{Var}(X_t), \quad T > 0, \quad t \in \mathbb{T}.$$

Acknowledgements

We are grateful to the referee for several valuable comments and suggestions, in particular for calling our attention to the examples presented in Section 3.

References

- [1] J. BERAN, Y. FENG, S. GHOSH and R. KULIK (2013). *Long-Memory Processes, Probabilistic Properties and Statistical Methods*. Springer-Verlag Berlin Heidelberg.
- [2] H. BIERMÉ, A. ESTRADE and I. KAJ, Self-similar random fields and rescaled random balls models, *J. Theor. Probab.* **23** (2010), 1110–1141.
- [3] H. BIERMÉ, M. M. MEERSCHAERT and H.-P. SCHEFFLER, Operator scaling stable random fields, *Stoch. Process. Appl.* **117** (2007), 312–332.
- [4] T. BOJDECKI, L. G. GOROSTIZA and A. TALARCZYK, Sub-fractional Brownian motion and its relation to occupation times, *Statistics & Probability Letters* **69** (2004), 405–419.
- [5] R. GAIGALAS, A Poisson bridge between fractional Brownian motion and stable Lévy motion, *Stoch. Process. Appl.* **116** (2006), 447–462.
- [6] R. GAIGALAS and I. KAJ, Convergence of scaled renewal processes and a packet arrival model, *Bernoulli* **9** (2003), 671–703.
- [7] E. IGLÓI, *Dilative Stability*, Ph.D. Thesis, University of Debrecen, Faculty of Informatics, 2008. URL: <http://www.inf.unideb.hu/valseg/dolgozok/igloi/dissertation.pdf>
- [8] E. IGLÓI and M. BARCZY, Path properties of dilatively stable processes and singularity of their distributions, *Stochastic Analysis and Applications* **30**(5) (2012), 831–848.
- [9] I. KAJ, Limiting fractal random processes in heavy-tailed systems. In: *Fractals in Engineering, New Trends in Theory and Applications*, 199–218. Eds. J. Levy-Lehel, E. Lutton, Springer-Verlag, London, 2005.

- [10] I. KAJ and A. MARTIN-LÖF, Scaling limit results for the sum of many inverse Lévy subordinators, Preprint (2012), URL: <http://arxiv.org/abs/1203.6831>
- [11] I. KAJ and M. S. TAQQU, Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In: *In and Out of Equilibrium 2, Progress in Probability* **60** (2008), 383–427. Eds. M. E. Vares, V. Sidoravicius, Birkhäuser, Basel.
- [12] C. KLÜPPELBERG and M. MATSUI, Generalized fractional Lévy processes with fractional Brownian motion limit, (2014).
URL: <http://mediatum.ub.tum.de/doc/1156351/1156351.pdf>
- [13] J. W. LAMPERTI, Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** (1962), 62–78.
- [14] T. MARQUARDT, Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli* **12** (2006), 1099–1126.
- [15] V. PILIPUSKAITĖ and D. SURGAILIS, Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes, *Stoch. Process. Appl.* **124** (2014), 1011–1035.
- [16] G. SAMORODNITSKY and M. S. TAQQU (1994). *Stable non-Gaussian Random Processes. Stochastic Models with Infinite Variance*. Chapman & Hall, New York.
- [17] A. SGHIR, The generalized sub-fractional Brownian motion. *Communications on Stochastic Analysis* **7**(3) (2013), 373–382.
- [18] D. W. STROOCK (1993). *Probability Theory, An Analytic View*. Cambridge University Press, Cambridge.
- [19] C. A. TUDOR (2013). *Analysis of Variations for Self-Similar Processes*. Springer-Verlag, Berlin.